Supplementary information for

Anomalous Maxwell-Garnett theory for photonic time crystals

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Guide To the Supplementary Sections

S1 Accuracy of Maxwell-Garnett theory in predicting the transmission and reflection coefficients of photonic time crystal

In this section, we analytically prove the accuracy of various Maxwell-Garnett theories, namely equations (4), (6), and (8) in the main text, in predicting the transmission and reflection coefficients of photonic time crystal; specifically, we show their equivalence with those in the effective temporal slab.

S1.1 General formulation for space-harmonic fields

In this subsection, we start with the electromagnetic fields of a particular wavevector k (e.g., electric displacement $\boldsymbol{D}_k$ and magnetic flux density $\boldsymbol{B}_k$) in the steady state, namely space-harmonic fields [79]. On this basis, Fourier theory can be applied to study the space-domain fields as follows

$$\begin{aligned} B(z,t) &= \int_{-\infty}^{+\infty} dk\ B_k(t)\cdot e^{ikz} \\ D(z,t) &= \int_{-\infty}^{+\infty} dk\ D_k(t)\cdot e^{ikz} \end{aligned} \tag{S1}$$

where we only consider a one-dimensional space r for conceptual brevity.

For the photonic time crystal with the structural setup in Fig. 1 in the main text, the permittivity and permeability in the whole space-time domain are given by

$$\varepsilon(t)=\varepsilon_j,\quad \mu(t)=\mu_j,\quad \text{region } j,\ 1\le j\le N+1 \tag{S2}$$

where N is the total temporal interface number; $\varepsilon_{\mathrm{MG}}$ and μ_{MG} are the homogenized effective parameters. The field expressions for $\boldsymbol{D}_k$ and $\boldsymbol{B}_k$ (the subscript k is neglected for concise expression) are assumed as follows

$$\begin{aligned} B(t) &= \begin{cases} a_1^+ e^{-i\omega_j(t-t_1)} & t\le t_1 \text{ (region 1)} \\ a_j^+ e^{-i\omega_j\left(t-t_{j-1}\right)} + a_j^- e^{+i\omega_j\left(t-t_{j-1}\right)} & t_{j-1}<t\le t_j \text{ (region } j) \\ a_{N+1}^+ e^{-i\omega_j(t-t_N)} + a_{N+1}^- e^{+i\omega_j(t-t_N)} & t_N<t \text{ (region } N+1) \end{cases} \\ D(t) &= \begin{cases} -\dfrac{1}{\eta_1} a_1^+ e^{-i\omega_j(t-t_1)} & t\le t_1 \\ -\dfrac{1}{\eta_j} a_j^+ e^{-i\omega_j\left(t-t_{j-1}\right)} + \dfrac{1}{\eta_j} a_j^- e^{+i\omega_j\left(t-t_{j-1}\right)} & t_{j-1}<t\le t_j \\ -\dfrac{1}{\eta_{N+1}} a_{N+1}^+ e^{-i\omega_j(t-t_N)} + \dfrac{1}{\eta_{N+1}} a_{N+1}^- e^{+i\omega_j(t-t_N)} & t_N<t \end{cases} \end{aligned} \tag{S3}$$

where a_j^+ (a_j^-) is the amplitude of the forward (backward) propagating wave components, and η_j and ω_j are the wave impedance and wave frequency given by

$$\eta_j=\frac{\mu_j\omega_j}{k}=\sqrt{\mu_j/\varepsilon_j}\ ,\quad \omega_j=k\Big/\sqrt{\mu_j\varepsilon_j}\ ,\quad \forall j \tag{S4}$$

S1.2 General formulation for the temporal characteristic matrix

In this subsection, we derive the characteristic matrix $\overline{\overline{M}}_j$ for a single temporal slab extending from $t = t_{j-1}$ to $t = t_j$, which relates the field values at its two temporal interfaces, namely

$$\begin{bmatrix} B_j(t_j) \\ D_j(t_j) \end{bmatrix} = \overline{\overline{M}}_j \begin{bmatrix} B_{j-1}(t_{j-1}) \\ D_{j-1}(t_{j-1}) \end{bmatrix} \tag{S5}$$

Equation (S5) is the generalization of Born's formulation for a single spatial slab [80] into the temporal case. The solution to $\overline{\overline{M}}_j$ can be obtained by enforcing temporal boundary condition and simple geometric optics. One the one hand, the continuity of the electric displacement $\boldsymbol{D}$ and magnetic flux density $\boldsymbol{B}$ before and after the temporal interface should be guaranteed, namely

$$\begin{bmatrix} B_{j-1}(t_{j-1}) \\ D_{j-1}(t_{j-1}) \end{bmatrix} = \begin{bmatrix} B_j(t_{j-1}) \\ D_j(t_{j-1}) \end{bmatrix}, \quad \forall j \in [2, N+1] \tag{S6}$$

Also note that the wavevector k is a conservable quantity due to the boundary condition.

On the other hand, from the perspective of geometric optics by following equation (S3), one has

$$\begin{gathered} B_j(t_j) = e^{-i\omega_j\tau_j} a_j^+ + e^{+i\omega_j\tau_j} a_j^- \\ D_j(t_j) = -\frac{1}{\eta_j} e^{-i\omega_j\tau_j} a_j^+ + \frac{1}{\eta_j} e^{+i\omega_j\tau_j} a_j^- \\ \tau_j = t_j - t_{j-1} \end{gathered} \tag{S7}$$

where τ_j is the temporal duration of the slab in region j. The amplitudes a_j^+ and a_j^- can be obtained by setting $t = t_{j-1}$ in equation (S3), and are related to $\boldsymbol{B}(t_j)$ and $\boldsymbol{D}(t_{j-1})$ by

$$\begin{bmatrix} a_j^+ \\ a_j^- \end{bmatrix} = \begin{bmatrix} 1/2 & -\eta_j/2 \\ 1/2 & \eta_j/2 \end{bmatrix} \begin{bmatrix} B_j(t_{j-1}) \\ D_j(t_{j-1}) \end{bmatrix} \tag{S8}$$

One can also write equation (S8) equivalently as

$$\begin{bmatrix} B_j(t_{j-1}) \\ D_j(t_{j-1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1/\eta_j & 1/\eta_j \end{bmatrix} \begin{bmatrix} a_j^+ \\ a_j^- \end{bmatrix} \tag{S9}$$

By substituting equation (S8) into equation (S7), and after some algebra, one has

$$\begin{bmatrix} B_j(t_j) \\ D_j(t_j) \end{bmatrix} = \begin{bmatrix} \cos(\omega_j\tau_j) & i\eta_j \sin(\omega_j\tau_j) \\ i\sin(\omega_j\tau_j)/\eta_j & \cos(\omega_j\tau_j) \end{bmatrix} \begin{bmatrix} B_j(t_{j-1}) \\ D_j(t_{j-1}) \end{bmatrix}, \quad \forall j \in [2, N+1] \tag{S10}$$

By combining formulas (S6) and (S10), one has the expression for temporal characteristic matrix as follows

$$\overline{\overline{M}}_j = \begin{bmatrix} \cos(\omega_j\tau_j) & i\eta_j \sin(\omega_j\tau_j) \\ i\sin(\omega_j\tau_j)/\eta_j & \cos(\omega_j\tau_j) \end{bmatrix} \tag{S11}$$

S1.3 Temporal characteristic matrix for the photonic time crystal and the homogenized temporal slab

In this subsection, we obtain the temporal characteristic matrix $\overline{\overline{M}}_{\mathrm{PTC}}$ for the photonic time crystal and the homogenized temporal slab. For the photonic time crystal, we start with its unit-cell characteristic matrix as follows

$$\begin{aligned}\overline{\overline{M}}_{\mathrm{PTC,unit}} &= \begin{bmatrix} \overline{\overline{M}}_{\mathrm{PTC,unit,11}} & \overline{\overline{M}}_{\mathrm{PTC,unit,12}} \\ \overline{\overline{M}}_{\mathrm{PTC,unit,21}} & \overline{\overline{M}}_{\mathrm{PTC,unit,22}} \end{bmatrix} \\ &= \overline{\overline{M}}_{\mathrm{I}}\overline{\overline{M}}_{\mathrm{II}} = \begin{bmatrix} \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}}) & i\eta_{\mathrm{I}}\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}}) \\ i\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}})/\eta_{\mathrm{I}} & \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}}) \end{bmatrix} \begin{bmatrix} \cos(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) & i\eta_{\mathrm{II}}\sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) \\ i\sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}})/\eta_{\mathrm{II}} & \cos(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) \end{bmatrix}\end{aligned} \tag{S12}$$

After some algebra, one has

$$\begin{aligned}\overline{\overline{M}}_{\mathrm{PTC,unit,11}} &= \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}})\cos(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) - \sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}})\sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}})\,\eta_{\mathrm{I}}/\eta_{\mathrm{II}} \\ \overline{\overline{M}}_{\mathrm{PTC,unit,12}} &= i\eta_{\mathrm{II}}\cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}})\sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) + i\eta_{\mathrm{I}}\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}})\cos(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) \\ \overline{\overline{M}}_{\mathrm{PTC,unit,21}} &= i/\eta_{\mathrm{I}}\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}})\cos(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) + i/\eta_{\mathrm{II}}\cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}})\sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) \\ \overline{\overline{M}}_{\mathrm{PTC,unit,22}} &= \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}})\cos(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) - \sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}})\sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}})\,\eta_{\mathrm{II}}/\eta_{\mathrm{I}}\end{aligned} \tag{S13}$$

Note that $\overline{\overline{M}}_{\mathrm{PTC,unit}}$ is a unimodular matrix, then, with some knowledge from the matrix theory [80], one has

$$\begin{aligned}\overline{\overline{M}}_{\mathrm{PTC}} = \overline{\overline{M}}_{\mathrm{PTC,unit}}^{N_{\mathrm{unit}}} &= \begin{bmatrix} \overline{\overline{M}}_{\mathrm{PTC,unit,11}}U_{N_{\mathrm{unit}}-1}(a) - U_{N_{\mathrm{unit}}-2}(a) & \overline{\overline{M}}_{\mathrm{PTC,unit,12}}U_{N_{\mathrm{unit}}-1}(a) \\ \overline{\overline{M}}_{\mathrm{PTC,unit,21}}U_{N_{\mathrm{unit}}-1}(a) & \overline{\overline{M}}_{\mathrm{PTC,unit,22}}U_{N_{\mathrm{unit}}-1}(a) - U_{N_{\mathrm{unit}}-2}(a) \end{bmatrix} \\ a &= \frac{\overline{\overline{M}}_{\mathrm{PTC,unit,11}} + \overline{\overline{M}}_{\mathrm{PTC,unit,21}}}{2} = \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}})\end{aligned} \tag{S14}$$

where $U_{N_{\mathrm{unit}}}(\cos x) = \sin\left[(N_{\mathrm{unit}}+1)x\right]/\sin x$ represents the Chebyshev polynomials of the second kind.

For the homogenized temporal slab, it is easy to write its characteristic matrix as follows based on the derivation in last subsection.

$$\begin{aligned}\overline{\overline{M}}_{\mathrm{MG}} &= \begin{bmatrix} \cos(\omega_{\mathrm{MG}}\tau_{\mathrm{MG}}) & i\eta_{\mathrm{MG}}\sin(\omega_{\mathrm{MG}}\tau_{\mathrm{MG}}) \\ i\sin(\omega_{\mathrm{MG}}\tau_{\mathrm{MG}})/\eta_{\mathrm{MG}} & \cos(\omega_{\mathrm{MG}}\tau_{\mathrm{MG}}) \end{bmatrix} \\ \eta_{\mathrm{MG}} = \sqrt{\mu_{\mathrm{MG}}/\varepsilon_{\mathrm{MG}}}\,&,\quad \omega_{\mathrm{MG}} = k/\sqrt{\mu_{\mathrm{MG}}\varepsilon_{\mathrm{MG}}}\,,\quad \tau_{\mathrm{MG}} = (\tau_{\mathrm{I}} + \tau_{\mathrm{II}})\cdot N_{\mathrm{unit}}\end{aligned} \tag{S15}$$

where τ_{MG}, ω_{MG}, η_{MG} are the effective temporal duration, angular frequency and impedance of the temporal slab obtained via the Maxwell-Garnett theory.

S1.4 Transmission and reflection coefficients, and the energy transmittivity and reflectivity

In this subsection, we give a general derivation for the transmission and reflection coefficients $\widetilde{t}$ and $\widetilde{r}$, and the energy transmittivity and reflectivity for the photonic time crystal and the homogenized temporal slab.

For the photonic time crystal, $\widetilde{t}_{\mathrm{PTC}}$ and $\widetilde{r}_{\mathrm{PTC}}$ are defined as

$$\begin{aligned}\widetilde{t}_{\mathrm{PTC}} &= a_{N+1}^{+}/a_1^{+}\\ \widetilde{r}_{\mathrm{PTC}} &= a_{N+1}^{-}/a_1^{+}\end{aligned} \tag{S16}$$

Note here the transmission and reflection coefficients are defined with expect to the magnetic flux density $\boldsymbol{B}$. By the definition of the characteristic matrixes for equation (S5) and by combing equations (S8-S9), one has

$$\begin{bmatrix}1 & 1\\ -1/\eta_{N+1} & 1/\eta_{N+1}\end{bmatrix}\begin{bmatrix}a_{N+1}^{+}\\ a_{N+1}^{-}\end{bmatrix} = \overline{\overline{M}}_{\mathrm{PTC}}\begin{bmatrix}1 & 1\\ -1/\eta_1 & 1/\eta_1\end{bmatrix}\begin{bmatrix}a_1^{+}\\ a_1^{-}\end{bmatrix} \tag{S17}$$

After some calculation, one has the scattering matrix $\overline{\overline{S}}_{\mathrm{PTC}}$ for the photonic time crystal, namely

$$\begin{aligned}\begin{bmatrix}a_{N+1}^{+}\\ a_{N+1}^{-}\end{bmatrix} &= \overline{\overline{S}}_{\mathrm{PTC}}\begin{bmatrix}a_1^{+}\\ a_1^{-}\end{bmatrix}\\ \overline{\overline{S}}_{\mathrm{PTC}} &= \begin{bmatrix}1/2 & -\eta_{N+1}/2\\ 1/2 & \eta_{N+1}/2\end{bmatrix}\overline{\overline{M}}_{\mathrm{PTC}}\begin{bmatrix}1 & 1\\ -1/\eta_1 & 1/\eta_1\end{bmatrix}\end{aligned} \tag{S18}$$

Using the fact that $a_1^{-} = 0$ for incident light, then one has

$$\begin{aligned}\widetilde{t}_{\mathrm{PTC}} = \overline{\overline{S}}_{\mathrm{PTC},11} &= \frac{1}{2}\left(\overline{\overline{M}}_{\mathrm{PTC},11} - \overline{\overline{M}}_{\mathrm{PTC},12}/\eta_1\right) - \frac{\eta_{N+1}}{2}\left(\overline{\overline{M}}_{\mathrm{PTC},21} - \overline{\overline{M}}_{\mathrm{PTC},22}/\eta_1\right)\\ \widetilde{r}_{\mathrm{PTC}} = \overline{\overline{S}}_{\mathrm{PTC},21} &= \frac{1}{2}\left(\overline{\overline{M}}_{\mathrm{PTC},11} - \overline{\overline{M}}_{\mathrm{PTC},12}/\eta_1\right) + \frac{\eta_{N+1}}{2}\left(\overline{\overline{M}}_{\mathrm{PTC},21} - \overline{\overline{M}}_{\mathrm{PTC},22}/\eta_1\right)\end{aligned} \tag{S19}$$

By following the same procedure, one has the transmission and reflection coefficients (i.e. $\widetilde{t}_{\mathrm{MG}}$ and $\widetilde{r}_{\mathrm{MG}}$) for the homogenized temporal slab, namely

$$\begin{aligned}\widetilde{t}_{\mathrm{MG}} &= \frac{1}{2}\left(\overline{\overline{M}}_{\mathrm{MG},11} - \overline{\overline{M}}_{\mathrm{MG},12}/\eta_1\right) - \frac{\eta_{N+1}}{2}\left(\overline{\overline{M}}_{\mathrm{MG},21} - \overline{\overline{M}}_{\mathrm{MG},22}/\eta_1\right)\\ \widetilde{r}_{\mathrm{MG}} &= \frac{1}{2}\left(\overline{\overline{M}}_{\mathrm{MG},11} - \overline{\overline{M}}_{\mathrm{MG},12}/\eta_1\right) + \frac{\eta_{N+1}}{2}\left(\overline{\overline{M}}_{\mathrm{MG},21} - \overline{\overline{M}}_{\mathrm{MG},22}/\eta_1\right)\end{aligned} \tag{S20}$$

On this basis, we obtain the energy transmittivity $\widetilde{T}$ and $\widetilde{R}$ reflectivity. By using the complex Poynting's theorem, the complex Poynting's vector for the incident wave is given by

$$\overline{S}_i = \frac{1}{2}Re\left[\overline{E}_1(t)\times\overline{H}_1^{*}(t)\right] = \hat{k}\frac{\left|a_1^{+}\right|^2}{2\mu_1\sqrt{\varepsilon_1\mu_1}} \tag{S21}$$

where $\hat{k}$ is the unit vector in the direction of the wavevector $\overline{k}$. Similarly, one has the complex Poynting's vector for the transmitted and reflected wave as follows

$$\overline{S}_t = \hat{k}\frac{\left|a_{N+1}^{-}\right|^2}{2\mu_{N+1}\sqrt{\varepsilon_{N+1}\mu_{N+1}}}$$
$$\overline{S}_r = \hat{k}\frac{\left|a_{N+1}^{-}\right|^2}{2\mu_{N+1}\sqrt{\varepsilon_{N+1}\mu_{N+1}}} \tag{S22}$$

Therefore, the energy transmittivity $\widetilde{T}$ and reflectivity $\widetilde{R}$ are related to the transmission and reflection coefficients ($\widetilde{t}$ and $\widetilde{r}$) by

$$\widetilde{T} = \frac{\mu_1\sqrt{\varepsilon_1\mu_1}}{\mu_{N+1}\sqrt{\varepsilon_{N+1}\mu_{N+1}}}\left|\widetilde{t}\right|^2$$
$$\widetilde{R} = \frac{\mu_1\sqrt{\varepsilon_1\mu_1}}{\mu_{N+1}\sqrt{\varepsilon_{N+1}\mu_{N+1}}}\left|\widetilde{r}\right|^2 \tag{S23}$$

S1.5 Equivalence of the characteristic matrixes between the photonic time crystal and the effective temporal slab

Finally in this subsection, we prove the validity of various Maxwell-Garnett theory, by showing the equivalence of the transmission and reflection coefficients, between the photonic time crystals and their homogenized counterparts, namely

$$\widetilde{t}_{\text{PTC}} = \widetilde{t}_{\text{MG}} \text{ and } \widetilde{r}_{\text{PTC}} = \widetilde{r}_{\text{MG}} \tag{S24}$$

In light of equations (S19) and (S20), it is sufficient to prove equation (S24), if we can obtain

$$\overline{\overline{M}}_{\text{PTC}} = \overline{\overline{M}}_{\text{MG}} \tag{S25}$$

where $\overline{\overline{M}}_{\text{PTC}}$ and $\overline{\overline{M}}_{\text{MG}}$ are the characteristic matrixes for the photonic time crystal and the effective temporal slab, as respectively determined in equation (S14) and (S15). To satisfy equation (S25), one can reasonably expect a stricter condition in the periodic system, namely, the equivalence between the characteristic matrix $\overline{\overline{M}}_{\text{PTC,unit}}$ for each unit cell of the photonic time crystal and that ($\overline{\overline{M}}_{\text{MG,unit}}$) for the homogenized temporal slab of the same temporal duration, as follows

$$\overline{\overline{M}}_{\text{PTC,unit}} = \overline{\overline{M}}_{\text{MG,unit}} \tag{S26}$$

$$\begin{aligned}
\overline{\overline{M}}_{\text{PTC,unit,11}} &= \cos(\omega_{\text{I}}\tau_{\text{I}})\cos(\omega_{\text{II}}\tau_{\text{II}}) - \sin(\omega_{\text{I}}\tau_{\text{I}})\sin(\omega_{\text{II}}\tau_{\text{II}})\,\eta_{\text{I}}/\eta_{\text{II}} \\
\overline{\overline{M}}_{\text{PTC,unit,12}} &= i\eta_{\text{II}}\cos(\omega_{\text{I}}\tau_{\text{I}})\sin(\omega_{\text{II}}\tau_{\text{II}}) + i\eta_{\text{I}}\sin(\omega_{\text{I}}\tau_{\text{I}})\cos(\omega_{\text{II}}\tau_{\text{II}}) \\
\overline{\overline{M}}_{\text{PTC,unit,21}} &= i/\eta_{\text{I}}\sin(\omega_{\text{I}}\tau_{\text{I}})\cos(\omega_{\text{II}}\tau_{\text{II}}) + i/\eta_{\text{II}}\cos(\omega_{\text{I}}\tau_{\text{I}})\sin(\omega_{\text{II}}\tau_{\text{II}}) \\
\overline{\overline{M}}_{\text{PTC,unit,22}} &= \cos(\omega_{\text{I}}\tau_{\text{I}})\cos(\omega_{\text{II}}\tau_{\text{II}}) - \sin(\omega_{\text{I}}\tau_{\text{I}})\sin(\omega_{\text{II}}\tau_{\text{II}})\,\eta_{\text{II}}/\eta_{\text{I}}
\end{aligned} \tag{S27}$$

$$\overline{\overline{M}}_{\text{MG,unit}} = \begin{bmatrix} \cos\left(\omega_{\text{MG}}(\tau_{\text{I}} + \tau_{\text{II}})\right) & i\eta_{\text{MG}} \sin\left(\omega_{\text{MG}}(\tau_{\text{I}} + \tau_{\text{II}})\right) \\ i \sin\left(\omega_{\text{MG}}(\tau_{\text{I}} + \tau_{\text{II}})\right)/\eta_{\text{MG}} & \cos\left(\omega_{\text{MG}}(\tau_{\text{I}} + \tau_{\text{II}})\right) \end{bmatrix} \tag{S28}$$

Below, we show how equation (S26) is fulfilled under the condition of various Maxwell-Garnett theories.

For conventional type 1 of Maxwell-Garnett theory within the long-wavelength limit [81], as derived in equation (4) in the main text, namely

$$\begin{matrix} \dfrac{\tau_{\text{I}} + \tau_{\text{II}}}{\varepsilon_{\text{MG}}} = \dfrac{\tau_{\text{I}}}{\varepsilon_{\text{I}}} + \dfrac{\tau_{\text{II}}}{\varepsilon_{\text{II}}} \\ \dfrac{\tau_{\text{I}} + \tau_{\text{II}}}{\mu_{\text{MG}}} = \dfrac{\tau_{\text{I}}}{\mu_{\text{I}}} + \dfrac{\tau_{\text{II}}}{\mu_{\text{II}}} \end{matrix}, \text{ if } \textit{within} \text{ the long-wavelength limit (including } \omega_{\text{MG}}(\tau_{\text{I}} + \tau_{\text{II}}) \to 0) \tag{S29}$$

By using equation (S29), one can simplify η_{MG} and ω_{MG} as

$$\begin{gathered} \eta_{\text{MG}} = \sqrt{\frac{\mu_{\text{MG}}}{\varepsilon_{\text{MG}}}} = \sqrt{\frac{\dfrac{\tau_{\text{I}}}{\varepsilon_{\text{I}}} + \dfrac{\tau_{\text{II}}}{\varepsilon_{\text{II}}}}{\dfrac{\tau_{\text{I}}}{\mu_{\text{I}}} + \dfrac{\tau_{\text{II}}}{\mu_{\text{II}}}}} \\ \omega_{\text{MG}} = \frac{k}{\sqrt{\mu_{\text{MG}}\varepsilon_{\text{MG}}}} = \frac{k}{\tau_{\text{I}} + \tau_{\text{II}}} \sqrt{\left(\frac{\tau_{\text{I}}}{\varepsilon_{\text{I}}} + \frac{\tau_{\text{II}}}{\varepsilon_{\text{II}}}\right)\left(\frac{\tau_{\text{I}}}{\mu_{\text{I}}} + \frac{\tau_{\text{II}}}{\mu_{\text{II}}}\right)} \end{gathered} \tag{S30}$$

Moreover, within the long-wavelength limit, the characteristic matrixes are simplified to

$$\overline{\overline{M}}_{\text{PTC,unit}} = \begin{bmatrix} 1 & i\eta_{\text{II}}\omega_{\text{II}}\tau_{\text{II}} + i\eta_{\text{I}}\omega_{\text{I}}\tau_{\text{I}} \\ i\omega_{\text{I}}\tau_{\text{I}}/\eta_{\text{I}} + i\omega_{\text{II}}\tau_{\text{II}}/\eta_{\text{II}} & 1 \end{bmatrix} \tag{S31}$$

and

$$\overline{\overline{M}}_{\text{MG,unit}} = \begin{bmatrix} 1 & i\eta_{\text{MG}}\omega_{\text{MG}}(\tau_{\text{I}} + \tau_{\text{II}}) \\ i\omega_{\text{MG}}(\tau_{\text{I}} + \tau_{\text{II}})/\eta_{\text{MG}} & 1 \end{bmatrix} \tag{S32}$$

where the Taylor expansion of sine and cosine functions are used. Then, to prove $\overline{\overline{M}}_{\text{PTC,unit}} = \overline{\overline{M}}_{\text{MG,unit}}$ (or more accurately speaking $\overline{\overline{M}}_{\text{PTC,unit}} \approx \overline{\overline{M}}_{\text{MG,unit}}$ in this case), reduces to proving

$$\begin{cases} \eta_{\text{MG}}\omega_{\text{MG}}(\tau_{\text{I}} + \tau_{\text{II}}) = \eta_{\text{II}}\omega_{\text{II}}\tau_{\text{II}} + \eta_{\text{I}}\omega_{\text{I}}\tau_{\text{I}} \\ \omega_{\text{MG}}(\tau_{\text{I}} + \tau_{\text{II}})/\eta_{\text{MG}} = \omega_{\text{I}}\tau_{\text{I}}/\eta_{\text{I}} + \omega_{\text{II}}\tau_{\text{II}}/\eta_{\text{II}} \end{cases} \tag{S33}$$

At this point, equation (S33) can be easily derived through simple algebra based on equation (S30). The detailed mathematics are omitted here.

For anomalous type 2 of Maxwell-Garnett theory via impedance matching [82], as governed by equation (6) in the main text, namely

$$\begin{matrix} \dfrac{T_{\text{PTC}}}{\varepsilon_{\text{MG}}} = \dfrac{\tau_{\text{I}}}{\varepsilon_{\text{I}}} + \dfrac{\tau_{\text{II}}}{\varepsilon_{\text{II}}} \\ \dfrac{T_{\text{PTC}}}{\mu_{\text{MG}}} = \dfrac{\tau_{\text{I}}}{\mu_{\text{I}}} + \dfrac{\tau_{\text{II}}}{\mu_{\text{II}}} \end{matrix}, \text{ if } \eta_{\text{I}} = \eta_{\text{II}}, \text{ for } \forall\ \omega_{\text{MG}}T_{\text{PTC}}/2\pi = T_{\text{PTC}}/T_{\text{MG}} \tag{S34}$$

Based on equation (S34), one has

$$\eta_{\mathrm{MG}} = \eta_{\mathrm{I}} = \eta_{\mathrm{II}}$$
$$\omega_{\mathrm{MG}} = \frac{k}{\tau_{\mathrm{I}} + \tau_{\mathrm{II}}}\left(\frac{\tau_{\mathrm{I}}}{\sqrt{\mu_{\mathrm{I}}\varepsilon_{\mathrm{I}}}} + \frac{\tau_{\mathrm{II}}}{\sqrt{\mu_{\mathrm{II}}\varepsilon_{\mathrm{II}}}}\right) = \frac{\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}}{\tau_{\mathrm{I}} + \tau_{\mathrm{II}}} \tag{S35}$$

Moreover, based on the impedance matching condition, namely $\eta_{\mathrm{I}} = \eta_{\mathrm{II}}$, equations (S27) and (S28) respectively reduce to

$$\overline{\overline{M}}_{\mathrm{PTC,unit}} = \begin{bmatrix} \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) & i\eta_{\mathrm{I}}\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \\ i\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}})/\eta_{\mathrm{II}} & \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \end{bmatrix} \tag{S36}$$

and

$$\overline{\overline{M}}_{\mathrm{MG,unit}} = \begin{bmatrix} \cos\big(\omega_{\mathrm{MG}}(\tau_{\mathrm{I}} + \tau_{\mathrm{II}})\big) & i\eta_{\mathrm{MG}}\sin\big(\omega_{\mathrm{MG}}(\tau_{\mathrm{I}} + \tau_{\mathrm{II}})\big) \\ i\sin\big(\omega_{\mathrm{MG}}(\tau_{\mathrm{I}} + \tau_{\mathrm{II}})\big)/\eta_{\mathrm{MG}} & \cos\big(\omega_{\mathrm{MG}}(\tau_{\mathrm{I}} + \tau_{\mathrm{II}})\big) \end{bmatrix} \tag{S37}$$

At this point, the equivalence between equations (S36) and (S37) is clear by substituting equation (S35) into them.

For anomalous type 3 of Maxwell-Garnett theory via temporal Fabry-Pérot, as governed by equation (8) in the main text, namely

$$\begin{matrix} \dfrac{T_{\mathrm{PTC}}}{\varepsilon_{\mathrm{MG}}} = \dfrac{\tau_{\mathrm{I}}}{\varepsilon_{\mathrm{I}}\eta_{\mathrm{I}}/\eta_{\mathrm{II}}} + \dfrac{\tau_{\mathrm{II}}}{\varepsilon_{\mathrm{II}}} \\ \dfrac{T_{\mathrm{PTC}}}{\mu_{\mathrm{MG}}} = \dfrac{\tau_{\mathrm{I}}}{\mu_{\mathrm{I}}\eta_{\mathrm{II}}/\eta_{\mathrm{I}}} + \dfrac{\tau_{\mathrm{II}}}{\mu_{\mathrm{II}}} \end{matrix}, \text{ if } \sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}}) = 0, \text{ for } \omega_{\mathrm{MG}}T_{\mathrm{PTC}}/2\pi = T_{\mathrm{PTC}}/T_{\mathrm{MG}} > 1/2 \tag{S38}$$

Based on equation (S38), one has

$$\eta_{\mathrm{MG}} = \eta_{\mathrm{II}}$$
$$\omega_{\mathrm{MG}} = \frac{k}{\tau_{\mathrm{I}} + \tau_{\mathrm{II}}}\left(\frac{\tau_{\mathrm{I}}}{\sqrt{\mu_{\mathrm{I}}\varepsilon_{\mathrm{I}}}} + \frac{\tau_{\mathrm{II}}}{\sqrt{\mu_{\mathrm{II}}\varepsilon_{\mathrm{II}}}}\right) = \frac{\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}}{\tau_{\mathrm{I}} + \tau_{\mathrm{II}}} \tag{S39}$$

Furthermore, based on the temporal Fabry-Pérot resonance condition, e.g. $\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}}) = 0$, equation (S27) reduces to

$$\begin{aligned} \overline{\overline{M}}_{\mathrm{PTC,unit}} &= \begin{bmatrix} (-1)^m\cos(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) & i\eta_{\mathrm{II}}(-1)^m\sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) \\ i/\eta_{\mathrm{II}}\,(-1)^m\sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) & (-1)^m\cos(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) \end{bmatrix} \\ &= \begin{bmatrix} \cos(m\pi + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) & i\eta_{\mathrm{II}}\sin(m\pi + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \\ i\sin(m\pi + \omega_{\mathrm{II}}\tau_{\mathrm{II}})/\eta_{\mathrm{II}} & \cos(m\pi + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \end{bmatrix} \end{aligned} \tag{S40}$$

where the identities of $\cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}}) = (-1)^m$ and $(-1)^m\cos(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) = \cos(m\pi + \omega_{\mathrm{II}}\tau_{\mathrm{II}})$ are used. By use the Fabry- Pérot resonance condition again, namely $\omega_{\mathrm{I}}\tau_{\mathrm{I}} = m\pi$, one has

$$\overline{\overline{M}}_{\mathrm{PTC,unit}} = \begin{bmatrix} \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) & i\eta_{\mathrm{II}}\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \\ i\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}})/\eta_{\mathrm{II}} & \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \end{bmatrix} \tag{S41}$$

Similarly, at this point, the equivalence between the characteristic matrix for the unit cell of the photonic time crystal in equation (S40), and that of the homogenized temporal slab of the same thickness can be easily obtained, based on equation (S39).

From all above, we have prove the equivalence between the transmission coefficient $\tilde{t}_{\text{PTC}}$ (or the reflection coefficient $\tilde{r}_{\text{PTC}}$) for a temporally finitely-thick photonic time crystal and that ($\tilde{t}_{\text{MG}}$ or $\tilde{r}_{\text{MG}}$) for the effective temporal slab, namely $\tilde{t}_{\text{PTC}} = \tilde{t}_{\text{MG}}$ (or $\tilde{r}_{\text{PTC}} = \tilde{r}_{\text{MG}}$), in a strict manner, by showing the equivalence between their character matrixes.

S2 Spatiotemporal evolution of various wave packets interacting with photonic time crystals beyond the long-wavelength limit

In this section we give the rigorous expressions for the field distribution of various space-time wave packet interacting with the photonic time crystal beyond the long-wavelength limit. The incident wave packet takes the form

$$\begin{bmatrix} a_1^+ \\ a_1^- \end{bmatrix} = \begin{bmatrix} a(k) \\ 0 \end{bmatrix} \tag{S42}$$

where $a(k)$ is the wavevector-dependent amplitude of the space-harmonic wave packet. For example, for the continuous Gaussian-type waveform, $a(k) = e^{-k^2/2\sigma_k^2}$. On this basis, one can obtain the field amplitude in region j, for $\forall j \in [2, N+1]$.

$$\begin{bmatrix} a_j^+ \\ a_j^- \end{bmatrix} = \begin{bmatrix} 1/2 & -\eta_j/2 \\ 1/2 & \eta_j/2 \end{bmatrix} \cdot \left[\prod_{n=j-1}^{1} \overline{\overline{M}}_n \right] \cdot \begin{bmatrix} 1 & 1 \\ -1/\eta_1 & 1/\eta_1 \end{bmatrix} \begin{bmatrix} a(k) \\ 0 \end{bmatrix}, \quad \forall j \in [2, N+1] \tag{S43}$$

By substituting the values of a_j^+ and a_j^- into equation (S3), all the spatiotemporal evolution of the wave packet can be obtained.

S3 More discussion on anomalous Maxwell-Garnett theory for photonic time crystals.

S3.1 Supplementary case of photonic time crystals with both constituents satisfying Fabry-Pérot resonance condition

In this subsection, we compare results for cases where one or both constituents satisfy the Fabry-Pérot resonance condition.

<u>*Case 1*</u>: When only one constituent of photonic time crystal satisfies the Fabry-Pérot resonance condition, we have

$$\sin(\omega_{\text{I}}\tau_{\text{I}}) = \sin(m\pi) = 0, \quad \textit{but} \ \sin(\omega_{\text{II}}\tau_{\text{II}}) \neq 0 \tag{S44}$$

where $m \in \mathbb{N}$. Generally, the corresponding characteristic matrix of each constituent X (X = I or II) is $\overline{\overline{M}}_{\mathrm{X}} = \begin{bmatrix} \cos(\omega_{\mathrm{X}}\tau_{\mathrm{X}}) & i\eta_{\mathrm{I}}\sin(\omega_{\mathrm{X}}\tau_{\mathrm{X}}) \\ i\sin(\omega_{\mathrm{X}}\tau_{\mathrm{X}})/\eta_{\mathrm{X}} & \cos(\omega_{\mathrm{X}}\tau_{\mathrm{X}}) \end{bmatrix}$. By using equation (S44), we have the characteristic matrix $\overline{\overline{M}}_{\mathrm{PTC,unit}}$ of the unit cell of the photonic time crystal, namely

$$\overline{\overline{M}}_{\mathrm{PTC,unit}} = \overline{\overline{M}}_{\mathrm{I}}\overline{\overline{M}}_{\mathrm{II}} = \begin{bmatrix} \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) & i\eta_{\mathrm{II}}\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \\ i\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}})/\eta_{\mathrm{II}} & \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \end{bmatrix} \tag{S45}$$

Meanwhile, the characteristic matrix $\overline{\overline{M}}_{\mathrm{MG}}$ of the homogenized slab with the same time duration is given by

$$\overline{\overline{M}}_{\mathrm{MG}} = \begin{bmatrix} \cos(\omega_{\mathrm{MG}}T_{\mathrm{PTC}}) & i\eta_{\mathrm{MG}}\sin(\omega_{\mathrm{MG}}T_{\mathrm{PTC}}) \\ i\sin(\omega_{\mathrm{MG}}T_{\mathrm{PTC}})/\eta_{\mathrm{MG}} & \cos(\omega_{\mathrm{MG}}T_{\mathrm{PTC}}) \end{bmatrix} \tag{S46}$$

When the Maxwell-Garnett effective medium theory holds, we can enforce the equivalence of the characteristic matrices above, namely $\overline{\overline{M}}_{\mathrm{PTC,unit}} = \overline{\overline{M}}_{\mathrm{MG}}$, or

$$\begin{bmatrix} \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) & i\eta_{\mathrm{II}}\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \\ i\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}})/\eta_{\mathrm{II}} & \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \end{bmatrix} = \begin{bmatrix} \cos(\omega_{\mathrm{MG}}T_{\mathrm{PTC}}) & i\eta_{\mathrm{MG}}\sin(\omega_{\mathrm{MG}}T_{\mathrm{PTC}}) \\ i\sin(\omega_{\mathrm{MG}}T_{\mathrm{PTC}})/\eta_{\mathrm{MG}} & \cos(\omega_{\mathrm{MG}}T_{\mathrm{PTC}}) \end{bmatrix} \tag{S47}$$

From the equivalence of diagonal terms and the equivalence of off-diagonal terms in equation (S47), we have

$$\cos(\omega_{\mathrm{MG}}T_{\mathrm{PTC}}) = \cos(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \tag{S48}$$

$$\eta_{\mathrm{II}}\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) = \eta_{\mathrm{MG}}\sin(\omega_{\mathrm{MG}}T_{\mathrm{PTC}}), \quad \textit{and} \quad \sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}})/\eta_{\mathrm{II}} = \eta_{\mathrm{MG}}\sin(\omega_{\mathrm{MG}}T_{\mathrm{PTC}})/\eta_{\mathrm{MG}} \tag{S49}$$

From equation (S48), we could arrive at a simple solution after some calculation, namely

$$\frac{T_{\mathrm{PTC}}}{\sqrt{\mu_{\mathrm{MG}}\varepsilon_{\mathrm{MG}}}} = \frac{\tau_{\mathrm{I}}}{\sqrt{\mu_{\mathrm{I}}\varepsilon_{\mathrm{I}}}} + \frac{\tau_{\mathrm{II}}}{\sqrt{\mu_{\mathrm{II}}\varepsilon_{\mathrm{II}}}} \tag{S50}$$

which is identical to equation (7) in the main text. Moreover, since we could generally have $\sin(\omega_{\mathrm{MG}}T_{\mathrm{PTC}}) = \sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) \neq 0$ according to equations (S44,S48), equation (S49) always requires that

$$\eta_{\mathrm{MG}} = \eta_{\mathrm{II}} \tag{S51}$$

In other words, if $\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}}) = 0$, *but* $\sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) \neq 0$, we can obtain determined solutions of $\varepsilon_{\mathrm{MG}}$ and μ_{MG} [e.g. equation (8) in the main text] by combing equations (S50-S51), as shown in Fig. S1(a).

Case 2: When both constituents of photonic time crystal satisfy the Fabry-Pérot resonance condition, we have

$$\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}}) = \sin(m\pi) = 0, \quad \textit{and} \quad \sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) = \sin(n\pi) = 0 \tag{S52}$$

where $m, n \in \mathbb{N}$. Similarly, when the Maxwell-Garnett effective medium theory holds, we always have $\overline{\overline{M}}_{\mathrm{PTC,unit}} = \overline{\overline{M}}_{\mathrm{MG}}$. This way, equations (S47-S50) are also satisfied. However, since we always have

$\sin(\omega_{\mathrm{MG}} T_{\mathrm{PTC}}) = \sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}} + \omega_{\mathrm{II}}\tau_{\mathrm{II}}) = \sin\big((m+n)\pi\big) \equiv 0$ according to equation (S52), equation (S49) is always satisfied, and the condition in equation (S51) is thus not mandatory. In other words, if $\sin(\omega_{\mathrm{I}}\tau_{\mathrm{I}}) = 0$, *and* $\sin(\omega_{\mathrm{II}}\tau_{\mathrm{II}}) = 0$, $\varepsilon_{\mathrm{MG}}$ and μ_{MG} are only governed by equation (S50) and thus will have undetermined solutions, as shown in Fig. S1(b).

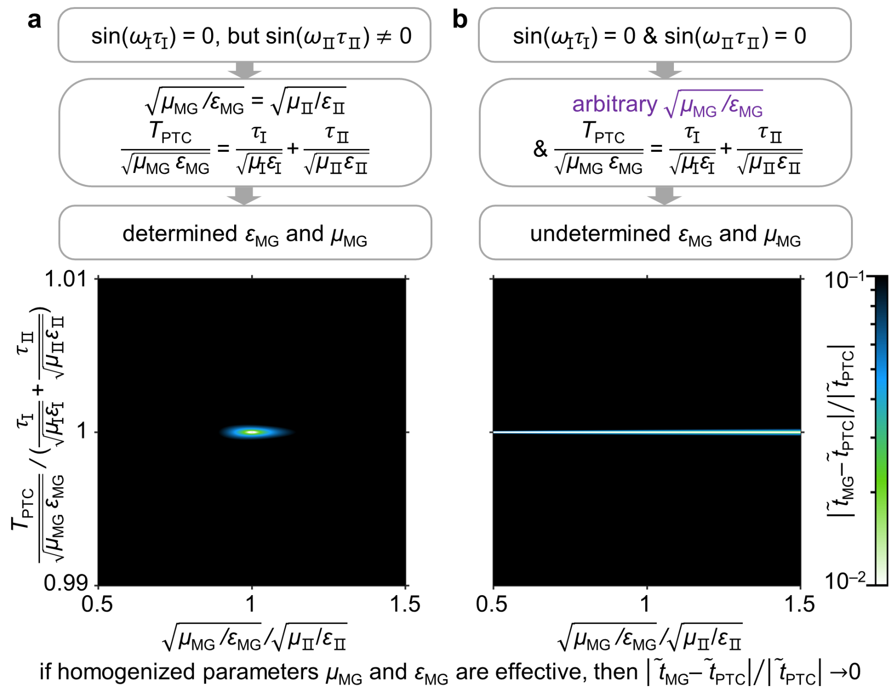

FIG. S1 Relative error in predicting the transmission coefficients of a real photonic time crystal ($\tilde{t}_{\mathrm{PTC}}$) via a homogenized temporal slab ($\tilde{t}_{\mathrm{MG}}$). The parameter space of interest is formed by $1/\sqrt{\mu_{\mathrm{MG}}\varepsilon_{\mathrm{MG}}}$ and $\sqrt{\mu_{\mathrm{MG}}\varepsilon_{\mathrm{MG}}}$, which can be easily transformed to $\varepsilon_{\mathrm{MG}}$ - μ_{MG} parameter space. The photonic time crystal has an interface number $N = 21$, $\varepsilon_{\mathrm{I}}/\varepsilon_0 = 1$, $\varepsilon_{\mathrm{II}}/\varepsilon_0 = 8.9$, $\mu_{\mathrm{I}}/\mu_0 = \mu_{\mathrm{II}}/\mu_0 = 1$, and $\omega_{\mathrm{I}}\tau_{\mathrm{I}} = \pi$. (a) A single constituent satisfies the Fabry-Pérot resonance condition. For illustration in (a), we set $\omega_{\mathrm{II}}\tau_{\mathrm{II}} = 9.15\pi$. (b) Both constituents satisfy the Fabry-Pérot resonance condition. In (b), $\omega_{\mathrm{II}}\tau_{\mathrm{II}} = 9\pi$. The solutions of $\varepsilon_{\mathrm{MG}}$ and μ_{MG} are governed by equations (S50,S51) and are determined in (a). In contrast, the solutions of $\varepsilon_{\mathrm{MG}}$ and μ_{MG} are only governed by equation (S50) and are thus undetermined in (b).

S3.2 Spatiotemporal evolution of light scattering in presence of arbitrary surrounding media

In this subsection, we show that our intension to set up the surrounding media with homogenized permittivity $\varepsilon_{\mathrm{MG}}$ and permeability μ_{MG}, is to ensure no light reflection at the interface between the surrounding environment and the real photonic time crystal. In this case, we can conveniently check the validity or accuracy of the Maxwell-Garnett theory by studying the transmission of light, as shown in Figs. S2(a) & S2(b) (namely Figs. 4(e) & 4(f) in the main text). Actually, its validity can also be checked without this specific setup of the surrounding media; see the example with the surrounding media being vacuum in Figs. S2(c) & S2(d).

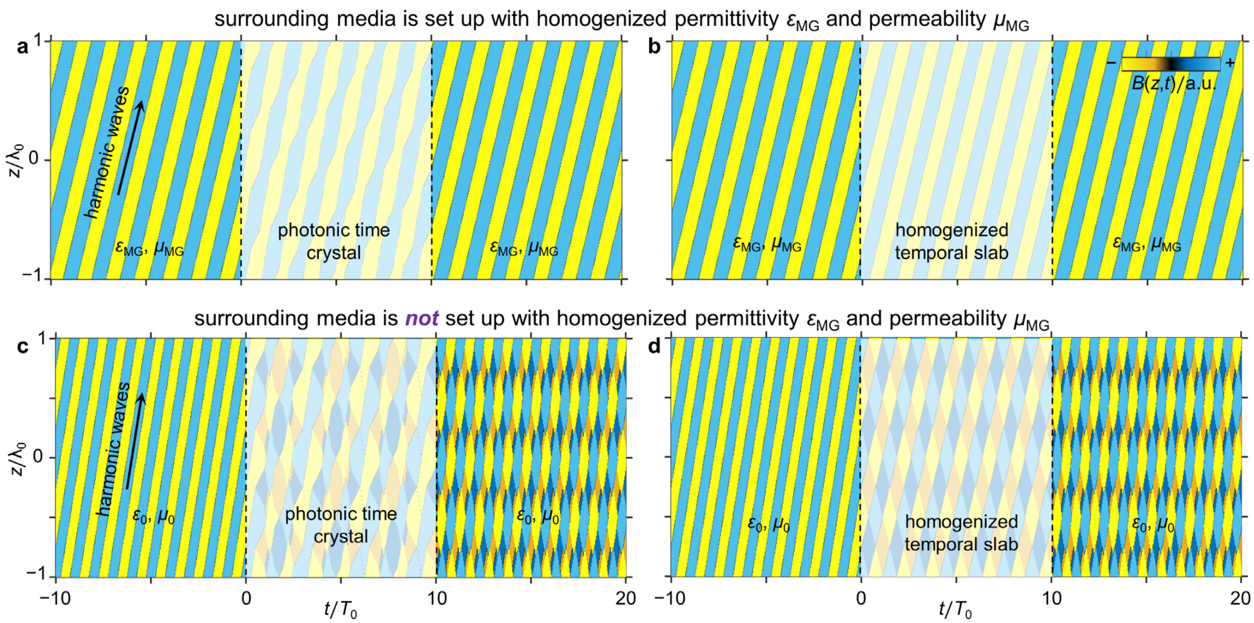

FIG. S2 Spatiotemporal evolution of space-time wave packets interacting with photonic time crystals in presence of different surrounding media. The setup in (a) & (c) for both the structures and the wave packets are the same as those in Fig. 4(e) in the main text, except that the surrounding media in (c) are vacuum. Similarly, (b) & (d) have the same set up as those in Fig. 4(f) in the main text, except that the surrounding media in (c) are vacuum. The perfectly same spatiotemporal evolution of light scattered by real and homogenized structures (e.g. $t/T_0 \in [10,20]$) in (a) & (b) or in (c) & (d) indicates the validity of Maxwell-Garnett theory.

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